MATH329: Final Exam (12 December, 1pm–3pm)

Name: ________________________________

Give clear and concise arguments for each of your claims.
A standard 8 1/2 by 11 sheet of paper with student's notes (both sides) is allowed.

1. Consider the matrix
$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}.$$
   (a) Find the reduced row echelon form of $\boldsymbol{A}$.
   (b) What is the dimension and a basis of the image of $\boldsymbol{A}$?
   (c) Find a basis for the kernel (nullspace) of $\boldsymbol{A}$.
   (d) If the vector $\boldsymbol{b}$ is the sum of the four columns of $\boldsymbol{A}$, write down a complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

2. Consider matrix
$$\boldsymbol{U} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$
   where all the constants non-zero.
   (a) Find the inverse of $\boldsymbol{U}$.
   (b) Suppose the columns of $\boldsymbol{U}$ are eigenvectors of a matrix $\boldsymbol{A}$. Show that $\boldsymbol{A}$ is also upper triangular.

3. True or false:
   (a) The rank of a matrix is equal to the number of its non-zero columns.
   (b) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.
   (c) Elementary row operations preserve rank.
   (d) Elementary column operations do not necessarily preserve rank.
   (e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix.
   (f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
   (g) The rank of an $n \times n$ matrix is at most n.
   (h) An $n \times n$ matrix having rank n is invertible.

4. Let $\langle \cdot \, , \cdot \rangle$ be the standard inner product in $\mathbf{C}^n$. Show that if for a matrix $\boldsymbol{A}$, $\boldsymbol{AA}^* = \boldsymbol{A}^*\boldsymbol{A}$ then
$$|\boldsymbol{Ax}| = |\boldsymbol{A}^*\boldsymbol{x}|$$
   for any $\boldsymbol{x} \in \mathbf{C}^n$. (Hint: Recall that $|\boldsymbol{x}|^2 = \langle \boldsymbol{x} \, , \boldsymbol{x} \rangle$ and use the matrix form for the standard inner product in $\mathbf{C}^n$.)

5. Prove that if matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar then $\det \boldsymbol{A} = \det \boldsymbol{B}$.

6. Let V be a vector space and let $v_1, \ldots, v_n$ be a basis in V. For $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and $y = \beta_1 v_1 + \ldots + \beta_n v_n$ *define* $\langle x \, , y \rangle = \sum_{j=1}^{n} \alpha_j \bar{\beta}_j$. Show that $\langle x \, , y \rangle$ defines an inner product in V.

7. Unitary diagonalize the following matrix:
$$\boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
   that is, find a unitary matrix $\boldsymbol{U}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{B} = \boldsymbol{UDU}^*$.